PMM U.S.S.R.,Vol.48,No.5,pp.556-562,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON THE THEORY OF A THREE-DIMENSIONAL, VISCOUS HYPERSONIC SHOCK LAYER IN THE NEIGHBOURHOOD OF THE PLANE OF SYMMETRY*

E.A. GERSHBEIN and S.A. YUNITSKII

A three-dimensional flow in a hypersonic viscous shock layer is studied near the plane of symmetry at Reynolds numbers ranging from the moderately small, to large. The solution of the system of equations of a shock layer /1/ is sought in the form of series in circular coordinates. The system of equations for the principal terms of the expansions appears not to be closed, since they contain terms with a longitudinal component of the peripheral pressure gradient which must be taken into account in order to obtain a correct description of the flow at large Reynolds numbers.

A procedure for truncating the series is proposed, enabling the system of equations to be closed for the principal terms of the expansion, as well as for the subsequent terms. The system of equations and boundary conditions obtained accurately describe the flow in the shock layer over the whole range of variation of the Reynolds numbers for which the equations of hypersonic viscous layer hold. A numerical solution of the problem is obtained over a wide range of variation of the Reynolds number, as well as the injection (suction) parameters. Characteristic profiles of the velocity and temperature components across the shock layer are given in various cross-sections of the plane of symmetry and also of the distributions of the pressure and heat transfer coefficient along the body surface.

The same difference mesh is used to obtain a solution for the equations of the three-dimensional boundary layer near the plane of symmetry on a permeable surface, and it is compared with the results of the theory of the viscous boundary layer. It is shown that the difference in the value of the coefficient of friction in the direction containing the cross-section of the body surface with the largest radius of curvature /1/ occurs not only in the neighbourhood of the stagnation point, but is retained along the whole of the plane of symmetry. The qualitative behaviour of the relation expressing the dependence of the ratio of the coefficients of friction obtained by solving the equations of the shock and boundary layers on the injection parameter at any point of the plane of symmetry, remains the same as at the stagnation point. The ratio increases with increasing suction and decreases with increasing injection, and at sufficiently large injection the values of the coefficient of friction agree at every point of the plane of symmetry.

A three-dimensional viscous shock layer near the plane of symmetry of bodies of revolution streamlined at the angle of attack, was studied in /2—6/ and various approaches were used to achieve the closure of the system of equations. In particular, in /2, 3/ the pressure gradient was approximated with help of the tables of inviscid supersonic flow, and in /4—6/ the pressure gradient in the peripheral direction was found by expanding the pressure in a trigonometric Fourier series in terms of the peripheral coordinate and retaining the first two terms of the expansion. A solution of the equations in the leeward part of the plane of symmetry of the blunt body was obtained in a form "tied" to the solution in the windward part. A numerical solution was obtained using the method of establishment for small angles of attack. Flow in a three-dimensional viscous shock layer, with a plane of symmetry, was also studied in /7, 8/.

At low Reynolds numbers, as the asymptotic analysis of the equations of the three-dimensional viscous shock layer implies /9/, terms containing an arbitrary component of the pressure gradient can be omitted, and the problem becomes closed for the principal terms near the plane of symmetry. Such an approach was developed in /10/, where a numerical solution was obtained for the equation of a hypersonic viscous shock layer near the plane of symmetry, and an analytic solution of the same problem was also obtained in the first-order approximation using the integral method of successive approximations /11/ for blunted bodies at varying angles of attack.

1. Three-dimensional hypersonic viscous shock layer near the plane of symmetry.

We shall consider three-dimensional flow past a blunt body in the coordinate

*Prikl.Matem.Mekhan.,48,5,768-775,1984

system normally attached to the streamlined body: x^1, x^2 are chosen on the body surface, and x^3 is orthogonal to the surface. The equations of a three-dimensional hypersonic viscous shock layer in a homogeneous gas, with the modified Rankine-Hugoniot relations at the shock wave when $x^3 = x_s^3$ and with boundary conditions on the body surface for $x^3 = 0$, are given in /1/. Let us change in Eqs.(1.1)–(1.3) of /1/ to the Dorodnitsyn variables

$$\xi \equiv \xi^1 = x^1, \quad \eta \equiv \xi^2 = x^2, \quad \zeta = \frac{1}{\Delta}\int_0^{x^3} \rho \sqrt{a}\, dx^3, \quad \Delta = \int_0^{x_s^3} \rho \sqrt{a}\, dx^3$$

($a = a_{11}a_{22} - a_{12}^2$, $a_{\alpha\beta}$ are the components of the first quadratic form of the body surface) and introduce two stream functions φ_1 and φ_2, so that the equation of continuity is satisfied identically

$$u[\alpha] = u^*_{(\alpha)}(\xi, \eta)\frac{\partial \varphi_\alpha}{\partial \zeta}, \quad u_\alpha^* = \varphi_\alpha^* \sqrt{a_{(\alpha\alpha)}}$$

$$\rho\sqrt{a}\, u[3] = -\frac{\partial}{\partial \xi^\alpha}(\Delta \varphi^*_{(\alpha)}\varphi_\alpha) - \Delta\varphi^*_{(\alpha)}\frac{\partial \varphi_\alpha}{\partial \zeta}\frac{\partial \zeta}{\partial x^\alpha}$$

$$T = T^*(\xi, \eta)\,\theta, \quad \partial\varphi_\alpha/\partial\zeta \equiv \varphi_\alpha' \equiv u_\alpha$$

Here and henceforth a prime denotes a derivative with respect to ζ, the Greek indices take the values 1, 2, repeated indices denote summation, and no summation is carried out over indices in the parenthesis, P, ρ, T, μ are the dimensionless pressure, density, temperature and viscosity of the gas respectively, $u[i]$ are the physical components of the velocity vector, σ, Re are the Prandtl and Reynolds numbers, γ is the ratio of the specific heats, and G is the specific, dimensionless gas flux through the body surface. The remaining symbols are defined in /1/. The function $u_\alpha^*(\xi, \eta)$, $T^*(\xi, \eta)$ will be defined below.

The equations of the three-dimensional hypersonic viscous shock layer can be written in the new variables in the form

$$(lu_\gamma')' + (B_\alpha\varphi_\alpha)u_\gamma' = \frac{\varepsilon}{\rho}E^{\gamma\alpha}\frac{\partial P}{\partial \xi^\alpha} + C^\gamma_{\alpha\beta}u_\alpha u_\beta + \tag{1.1}$$
$$D_\xi(\varphi_1, \varphi_\gamma) + D_\eta(\varphi_2, \varphi_\gamma), \quad \gamma = 1, 2$$

$$\left(\frac{l}{\sigma}\theta'\right)' + \left(B_\alpha\varphi_\alpha + \varphi^*_{(\alpha)}\frac{\partial \varphi_\alpha}{\partial \xi^\alpha}\right)\theta' = \frac{\varepsilon}{\rho}E^{3\alpha}u_{(\alpha)}\frac{\partial P}{\partial \xi^\alpha} +$$
$$lC^3_{\alpha\beta}u_\alpha' u_\beta' + \varphi^*_{(\alpha)}u_\alpha\left(\frac{\partial \theta}{\partial \xi^\alpha} + F_\alpha\theta\right)$$

$$P' = P_{\alpha\beta}^3 u_\alpha u_\beta, \quad P = \rho T^*\theta, \quad \mu = (T^*\theta)^\omega$$

where

$$D_\xi(a, b) \equiv \varphi_1^*\left(\frac{\partial a}{\partial \zeta}\frac{\partial^2 b}{\partial \xi\,\partial \zeta} - \frac{\partial a}{\partial \xi}\frac{\partial^2 b}{\partial \zeta^2}\right)$$

$$D_\eta(a, b) \equiv \varphi_2^*\left(\frac{\partial a}{\partial \zeta}\frac{\partial^2 b}{\partial \eta\,\partial \zeta} - \frac{\partial a}{\partial \eta}\frac{\partial^2 b}{\partial \zeta^2}\right)$$

$$l = \frac{\mu\rho a}{K\Delta^2}, \quad K = \varepsilon\,\mathrm{Re}, \quad \varepsilon = \frac{\gamma - 1}{2\gamma}, \quad \mathrm{Re} = \frac{\rho_\infty V_\infty R}{\mu_0}$$

$$\mu_0 = \mu(T_0), \quad T_0 = \frac{V_\infty^2}{2c_p}, \quad \gamma = \frac{c_p}{c_v}, \quad \sigma = \frac{\mu c_p}{\lambda}$$

The coefficients of the equations are

$$C^\alpha_{(\alpha\alpha)} = \varphi_\alpha^*\frac{\partial \ln u_\alpha^*}{\partial \xi^{(\alpha)}} + A^\alpha_{(\alpha\alpha)}u^*_{(\alpha)} \tag{1.2}$$

$$C^\gamma_{12} = C^\gamma_{21} = \frac{1}{2}\varphi^*_{(\alpha)}\frac{\partial \ln u_\gamma^*}{\partial \xi^\alpha} + A^\gamma_{12}u_\alpha^*$$

$$C^\gamma_{(\alpha\alpha)} = A^\gamma_{(\alpha\alpha)}\frac{(u_\alpha^*)^2}{u^*_{(\gamma)}}, \quad \alpha \neq \gamma; \quad B_\alpha = \frac{\partial \varphi^*_{(\alpha)}}{\partial \xi^\alpha} + \varphi^*_{(\alpha)}\frac{\partial \ln \Delta}{\partial \xi^\alpha}$$

$$E^{\gamma\alpha} = \frac{a^{\gamma\alpha}\sqrt{a_{(\gamma\gamma)}}}{u^*_{(\gamma)}}, \quad E^{3\alpha} = -\frac{2\varphi_\alpha^*}{T^*}, \quad F_\alpha = \frac{\partial \ln T^*}{\partial \xi^\alpha}$$

$$P_{\alpha\beta}^3 = A_{\alpha\beta}^3 u_{(\alpha)}^* u_{(\beta)}^*\Delta/\sqrt{a}$$

$$C_{\alpha\beta}^3 = -2a_{\alpha\beta}\varphi_{(\alpha)}^*\varphi_{(\beta)}^*/T^*$$

The boundary conditions take the following form in the Dorodnitsyn variables:

$$B_\alpha\varphi_\alpha + \varphi^*_{(\alpha)}\frac{\partial \varphi_\alpha}{\partial \xi^\alpha} = \Phi^{-1}, \quad P = v_\infty^2 \tag{1.3}$$

$$l\Phi u_\gamma' + u_\gamma - u_s(\gamma) = 0, \quad \gamma = 1, 2$$

$$l\Phi\sigma^{-1}\theta' + \theta - C_{\alpha\beta}{}^3 u_\alpha\,(l\Phi u_\beta' + {}^1/_2 u_\beta) = (T^*)^{-1}$$
$$\Phi^{-1} = -\frac{\sqrt{a}\,v_\infty}{\Delta}, \quad u_*(\gamma) \equiv \frac{u\,[\gamma]_\infty}{u^*_{(\gamma)}}, \quad v_\infty \equiv u\,[3]_\infty$$

on the shock wave when $\zeta = 1$ ($u\,[i]_\infty$ are the physical velocity components in the incoming flow written on the body surface), and

$$B_\alpha\varphi_\alpha + \varphi^*_{(\alpha)}\frac{\partial\varphi_\alpha}{\partial\xi^\alpha} = -\frac{G\sqrt{a}}{\Delta} \tag{1.4}$$
$$u_\gamma = 0, \quad \gamma = 1, 2; \quad \theta = \theta_w\,(\xi, \eta)$$

at the body surface when $\zeta = 0$.

In the first three equations of (1.1) terms proportional to $\partial P/\partial\zeta$ are omitted, since they are small in the part of the shock layer near the wall, compared with the remaining terms $\partial P/\partial\xi^\alpha$ [9–13].

An asymptotic analysis of the problem /9/ shows that the system (1.1)–(1.4) can be used with uniform success to describe flow in a hypersonic shock layer over a wide range of values of the Reynolds number and injection parameter.

Let us consider flow in a three-dimensional viscous shock layer on a smooth blunted body with a plane of symmetry. Let the surface of the body be described in a Cartesian system of coordinates by the equation $y^3 = f\,(y^1, y^2)$. We choose the following parametrization of the surface: $y^1 = \xi$, $y^2 = \eta$, $y^3 = f\,(\xi, \eta)$. Let $\xi = 0$ be the plane of symmetry of the flow, and let us write $u_\alpha{}^*\,(\xi, \eta) = u\,[\alpha]_\infty$, $T^*\,(\xi, \eta) = (u\,[3]_\infty)^2$. Assuming that the radius of transverse curvature of the body is finite, we obtain the following relations in the neighbourhood of the plane of symmetry:

$$u_1{}^* \sim \xi, \quad u_2{}^* \sim 1 + O\,(\xi^2), \quad T^* \sim 1 + O\,(\xi^2)$$

We shall seek the solution of system (1.1)–(1.4) near the plane of symmetry in the form

$$F\,(\xi, \eta, \zeta) = F_0\,(\eta, \zeta) + F_2\,(\eta, \zeta)\xi^2 + F_4\,(\eta, \zeta)\xi^4 + \ldots \tag{1.5}$$
$$P\,(\xi, \eta, \zeta) = P_0\,(\eta, \zeta) + P_2\,(\eta, \zeta)\xi^2 + P_4\,(\eta, \zeta)\xi^4 + \ldots$$

where F denotes anly of the functions $u_\gamma, \theta, l, \rho$; P is the pressure. Let us expand the coefficients of (1.2) in series in ξ, taking into account the parametrization and the choice of $u_\alpha{}^*, T^*$, and let us introduce the notation

$$C_{\alpha\beta}{}^l \equiv (C_{\alpha\beta}{}^k)_2, \quad l = k + 3, \quad k = 1, 2, 3 \tag{1.6}$$
$$P_{\alpha\beta}{}^4 \equiv (P_{\alpha\beta}{}^3)_2, \quad F_3 \equiv F_{10}, \quad \varphi_3{}^* \equiv \varphi_{10}{}^*$$
$$\varphi_4{}^* \equiv (\varphi_2{}^*)_2/(\varphi_2{}^*)_0, \quad B \equiv B_{\alpha 2}\varphi_\alpha + B_\alpha\varphi_{\alpha 2}$$

Then substituting the expansions (1.5) into (1.1)–(1.4) we obtain, taking into account the notation (1.6) (with the zero index omitted), the equations for the principal term

$$(lu_1')' + (B_\alpha\varphi_\alpha)\,u_1' = \frac{\varepsilon}{\rho}\Big(2E^{11}P_2 + E^{12}\frac{\partial P}{\partial\eta}\Big) + \tag{1.7}$$
$$C^1_{\alpha\beta}u_\alpha u_\beta + D_\eta(\varphi_2, \varphi_1)$$
$$(lu_2')' + (B_\alpha\varphi_\alpha)\,u_2' = \frac{\varepsilon}{\rho}E^{22}\frac{\partial P}{\partial\eta} + C^2_{22}\,(u_2)^2 + D_\eta(\varphi_2, \varphi_2)$$
$$\Big(\frac{l}{\sigma}\,\theta'\Big)' + \Big(B_\alpha\varphi_\alpha + \varphi_2{}^*\frac{\partial\varphi_2}{\partial\eta}\Big)\theta' = \frac{\varepsilon}{\rho}E^{22}u_2\frac{\partial P}{\partial\eta} +$$
$$lC^3_{22}\,(u_2')^2 + u_2D_T\theta$$
$$P' = P_{22}{}^3\,(u_2)^2, \quad P = \rho T^*\theta, \quad \mu = (T^*\theta)^\omega$$

and second term

$$(lu_{\gamma 2} + l_2u_\gamma')' + (B_\alpha\varphi_\alpha)\,u_{\gamma 2}' + Bu_\gamma' = \varepsilon R^\gamma + \tag{1.8}$$
$$2C^\gamma_{\alpha\beta}u_\alpha u_{\beta 2} + C^k_{\alpha\beta}u_\alpha u_\beta + 2D_\zeta\,(\varphi_{\gamma 2}, \varphi_\gamma) +$$
$$D_\eta\,(\varphi_2, \varphi_{\gamma 2}) + D_\eta\,(\varphi_{22}, \varphi_\gamma) + \varphi_4{}^*D_\eta\,(\varphi_2, \varphi_\gamma)$$
$$\gamma = 1, 2; \quad k = \gamma + 3$$
$$\Big(\frac{l}{\sigma}\,\theta_2' + \frac{l_2}{\sigma}\,\theta'\Big)' + \Big(B_\alpha\varphi_\alpha + \varphi_2{}^*\frac{\partial\varphi_2}{\partial\eta}\Big)\theta_2' +$$
$$\Big[B + 2\varphi_3{}^*\varphi_{12} + \varphi_2{}^*\Big(\frac{\partial\varphi_{22}}{\partial\eta} + \varphi_4{}^*\frac{\partial\varphi_2}{\partial\eta}\Big)\Big]\,\theta' = \varepsilon R^3 +$$
$$lC^6_{\alpha\beta}u_\alpha'u_\beta' + C^3_{22}u_2'\,(l_2u_2' + 2lu_{22}') +$$
$$2\varphi_3{}^*u_1\theta_2 + (\varphi_3{}^*F_3u_1 + \varphi^*_{22}u_2)\,\theta +$$
$$(u_{22} + \varphi_4{}^*u_2)\,D_T\theta + u_2D_T\theta_2$$
$$P_2' = P^4_{\alpha\beta}u_\alpha u_\beta + 2P^3_{22}u_2u_{22}$$
$$\frac{P_2}{P} = \frac{\rho_2}{\rho} + \frac{\theta_2}{\theta} + \frac{T_2{}^*}{T^*}$$

of the expansion.

Function R^1 has the form

$$R^1 = \frac{1}{\rho}\left[\left(E_2^{12} - \frac{\rho_2}{\rho}E^{12}\right)\frac{\partial P}{\partial \eta} + E^{12}\frac{\partial P_2}{\partial \eta} + 2\left(E_2^{11} - \frac{\rho_2}{\rho}E^{11}\right)P_2 + 4E^{11}P_4\right]$$

and the functions R^2 and R^3 can be written in the same form. They do not contain terms proportional to P_4.

The differential operators D_ζ, D_T act thus:

$$D_\zeta(a, b) \equiv \varphi_3^*\left(\frac{\partial \varphi_1}{\partial \zeta}\frac{\partial a}{\partial \zeta} - \varphi_{12}\frac{\partial^2 b}{\partial \zeta^2}\right)$$

$$D_T a \equiv \varphi_2^*\left(\frac{\partial a}{\partial \eta} + F_2 \cdot a\right)$$

The boundary conditions for the system of equations (1.7), (1.8) are

$$\begin{aligned}
&\zeta = 0: \quad B_\alpha\varphi_\alpha + \varphi_2^*\frac{\partial \varphi_2}{\partial \eta} = -\frac{G\sqrt{a}}{\Delta} \\
&u_\gamma = 0, \quad \gamma = 1, 2; \quad \theta = \theta_w(\eta) \\
&\zeta = 1: \quad B_\alpha\varphi_\alpha + \varphi_2^*\frac{\partial \varphi_2}{\partial \eta} = \Phi^{-1}, \quad P = v_\infty^2 \\
&l\Phi u_\gamma' + u_\gamma - u_s(\gamma) = 0 \\
&l\Phi\sigma^{-1}\theta' + \theta - C_{22}^3 u_2(l\Phi u_2' + {}^1/_2 u_2) = (T^*)^{-1}
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
&\zeta = 0: \quad u_{\gamma 2} = 0, \quad \gamma = 1, 2; \quad \theta_2 = \theta_{w2}(\eta) \\
&B + 2\varphi_3^*\varphi_{12} + \varphi_2^*\left(\frac{\partial \varphi_{22}}{\partial \eta} + \varphi_4^*\frac{\partial \varphi_2}{\partial \eta}\right) = -\frac{(G\sqrt{a})_2}{\Delta} + \frac{(G\sqrt{a})\Delta_2}{\Delta^2} \\
&\zeta = 1: \\
&B + 2\varphi_3^*\varphi_{12} + \varphi_2^*\left(\frac{\partial \varphi_{22}}{\partial \eta} + \varphi_4^*\frac{\partial \varphi_2}{\partial \eta}\right) = -\frac{\Phi_2}{\Phi^2} \\
&l\Phi u_{\gamma 2}' + u_{\gamma 2} + (l\Phi_2 + l_2\Phi)u_\gamma' - u_{s2}(\gamma) = 0 \\
&l\Phi\sigma^{-1}\theta_2' + \theta_2 + (l\Phi_2 + l_2\Phi)(\sigma^{-1}\theta' - C_{22}^3 u_2 u_2') - \\
&\quad l\Phi C_{22}^3(u_2 u_{22}' + u_{22}u_2') - C_{\alpha\beta}^6 u_\alpha(l\Phi u_\beta' + {}^1/_2 u_\beta) - \\
&\quad C_{22}^3 u_2 u_{22} = -T_2^*/(T^*)^2; \quad P_2 = 2v_\infty v_{2\infty}
\end{aligned} \tag{1.10}$$

(conditions (1.9) refer to the principal, and (1.10) to the second terms of the expansion).

We stress the fact that in the notation used the velocity component u_2 is directed along the plane of symmetry, and u_1 is, in fact, the velocity gradient in the peripheral ξ direction (the physical velocity component in this direction is identically equal to zero in the plane of symmetry).

The form of (1.7), (1.8) leads to the conclusion that the system of equations of the three-dimensional hypersonic viscous shock layer is not closed near the plane of symmetry for the principal, as well as for the subsequent terms of expansion (1.5). Indeed, system (1.7) contains the quantity P_2, which is the component of the pressure gradient in the peripheral direction and is found from the solution of system (1.8) which includes P_4, etc. Thus the system of equations (1.7) turns out to be connected with (1.8) through P_2, and the latter in turn is connected with the systems of equations for all subsequent terms of the expansion in terms of P_4.

It can be shown that near the stagnation point the problem is closed for the principal and subsequent terms of the expansion /1/, since the dependence of P_2 on the coefficients of the expansion of F_2 vanishes.

At low Reynolds numbers, when the terms with the longitudinal component of the pressure gradient can be neglected, the problem is closed near the plane of symmetry for the principal, as well as the remaining terms of expansion (1.5). The equations corresponding to this case are obtained by formally putting $\varepsilon \equiv 0$ in the right-hand sides of (1.7) and (1.8).

We propose the following procedure for truncating the series. A system of equations with boundary conditions (1.7)–(1.10) becomes closed when the term containing P_4 is omitted. In such an approach the terms of the expansion (1.5) F_0, P_0 and P_2 are determined near the plane of symmetry asymptotically correctly, by virtue of the hypersonic character of the flow $\varepsilon \to 0$, over the whole range of variation of the Reynolds number, from moderately small, to large. Indeed, when $\varepsilon \to 0$ and at low Reynolds numbers, the longitudinal components of the pressure gradient disappear from equations (1.7), (1.8). At large Reynolds numbers the effects of molecular transport appear in the boundary layer across which the pressure is constant. The pressure in the shock layer and at the body surface is found from Euler's equations containing no terms with the longitudinal components of the pressure gradient, and the transverse components of the pressure gradient, and the transverse component balanced by the centrifugal forces.

Note that the solution for the second terms of the expansion of F_2 obtained in this manner holds everywhere within the shock layer, except at the region adjacent to the wall, since the inclusion of P_4 becomes essential in this region for its determination.

Further truncations are performed in the same manner. In the system of equations for all F_3, F_4, P_4 we omit the term P_6, and F_3, P_4 are obtained asymptotically correctly.

We also note that the pressure is assumed to be known in problems of boundary layer theory. In this case P_2 is found from the solution of the outer inviscid problem, and the problem of boundary layer flow near the plane of symmetry is closed for the principal terms of expansion (1.5). For this reason, the problem of the closure of the system of equations for the three-dimensional boundary layer near the plane of symmetry /14/ does not arise.

2. Numerical solution of the equations of a three-dimensional viscous shock layer near the plane of symmetry on a permeable surface. When problem (1.7)–(1.10) was solved numerically, the terms R^j were neglected as non-ordinal. An explicit difference scheme /15/ with approximation accuracy of $O(\Delta\zeta^4, \Delta\eta^2)$ was used. The equations were solved one after another as they were written; the first-order equations for P_2 and $\eta^{-1}\partial P/\partial\eta$ were integrated from the surface of the shock wave to the body using Simpson's rule. Iterations over the functions $\Delta(\eta), \Delta_2(\eta)$ were carried out together with iterations of the system of finite difference equations necessary by virtue of its non-linearity. Every new value of the quantities Δ, Δ_2 was determined using a damping factor, usually equal to ½. There was no damping over the profiles.

An an example, we give the results of a calculation of the flow at zero angle of attack past an elliptic paraboloid the equation of whose surface in Cartesian coordinates has the form $2y^3 = (y^1)^2 + k(y^2)^2$ where $k = R_1/R_2$; R_1, R_2 are the principal radii of curvature of the surface at the tip of the paraboloid. The defining parameters of the problem were varied within the limits $0.1 \leqslant k \leqslant 1$, $1 \leqslant \mathrm{Re} \leqslant 10^6$, $0.5 \leqslant \omega \leqslant 1$, $-0.125 \leqslant G\sqrt{a} \leqslant 0.125$, $\varepsilon = 0.1$, $\sigma = 0.71$, $\theta_w(\eta) = 0.1$, $G\sqrt{a} = \mathrm{const}$.

Figs.1–4 show some of the results of the computations. The solid linees in Fig.1 depict the characteristic profiles of the tangential velocity components $u \equiv u_1, w \equiv u_2$ and the temperature θ across the shock layer when $\eta = 0$ (lines *1, 3*) and $\eta = 3$ (lines *2.4*) for two values of the Reynolds number, $\mathrm{Re} = 5$ (lines *1, 2*) and $\mathrm{Re} = 100$ (lines *3, 4*) and for $G = 0, k = 0.4, \omega = 0.5$. The distributions along the body surface of the pressure P_w, pressure gradient P_{2w} and the heat transfer coefficient referred to its values at the stagnation point q_r, are shown in Fig.2 by the solid lines for the same values of the parameters (lines *1* for $\mathrm{Re} = 5$, lines *2* for $\mathrm{Re} = 100$, the dot-dash lines depicting the pressure distribution over the body as given by Newton's formula). Here

$$q(\eta) = \frac{\lambda}{\rho_\infty V_\infty^3}\frac{\partial T}{\partial x^3}\Big|_{x^3=0} = T_* \frac{l\Delta}{2\sigma\sqrt{a}}\frac{\partial\theta}{\partial\zeta}\Big|_{\zeta=0}, \quad q_r = q(\eta)/q(0)$$

We note that when the Reynolds numbers exceed 100, the distributions of P_w and P_{2w} becomes practically independent of the further rise in the Reynolds number, and coincide with the lines *2*. When $\mathrm{Re} \simeq 1-5$, the distributions approach those given by Newton's formula. The influence of the longitudinal components of the pressure gradient on the flow parameters was determined by solving the system (1.7), (1.9) for the principal terms of the expansion from which the longitudinal pressure gradients were omitted /10/ (the dashed lines in Figs.1, 2). Computations show that when $5 \lesssim \mathrm{Re} \lesssim 100$, the influence of the omitted terms containing the pressure gradients on the distribution of the pressure and relative heat fluxes q_r along the body, is small. Thus, in the case of the conditions shown in Fig.2, the maximum deviation in the value of q_r obtained for various formulations of the problem does not exceed 5% at the point $\eta = 3$, $\mathrm{Re} = 100$. The difference in the absolute values of the coefficient q is in this case is $\sim 30\%$. The distributions in the values of the coefficient of friction behave in the same manner.

Fig.3 shows the profiles of u, w and θ in two cross-sections, $\eta = 0$ (lines *1, 3*) and $\eta = 3$ (lines *2, 4*) calculated for $\mathrm{Re} = 5\cdot 10^3$, $k = 0.4$, $\omega = 0.5$ and for two values of the specific gas flux across the surface $G\sqrt{a} = 0$ (lines *1, 2*) and $G\sqrt{a} = 0.125$ (lines *3, 4*). The graphs show that for an injection $G\sqrt{a} = 0.125$ the viscous boundary layer separates completely from the body along the whole plane of symmetry and becomes a displacement layer, and an inviscid layer of injected gas forms near the body. The maximum in the profiles of θ at $\eta = 3$ is explained by the fact that the stream lines situated near the body surface have passed through a strongly heated part of the shock layer near the stagnation point.

To compare the results obtained within the framework of the theory of the viscous shock layer and boundary layer, we solved numerically the system of equations of three-dimensional boundary layer near the plane of symmetry with injection or suction, using the same finite difference method /15/ and on the same difference meshes as system (1.7)–(1.10). The equations of the three-dimensional boundary layer near the plane of symmetry /14/, written in terms of the Dorodnitsyn variables, have the form (1.7), provided that the fourth equation is written in the form $P' = 0$. The boundary conditions on the outer boundary of the boundary layer are specified in the usual manner, and the numerical solution used the values of the

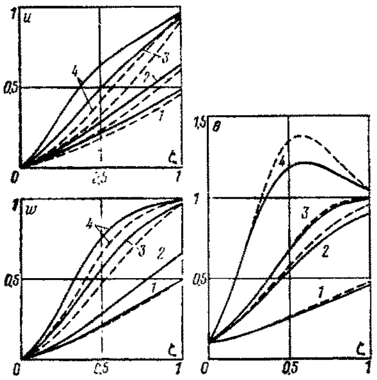

Fig.1

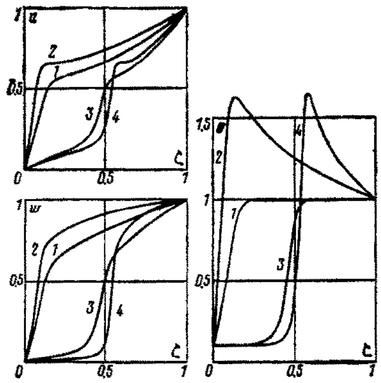

Fig.3

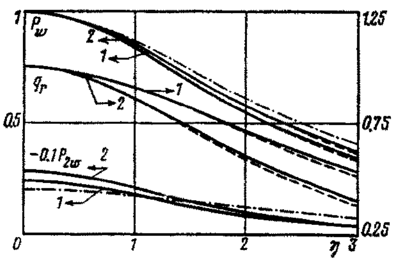

Fig.2

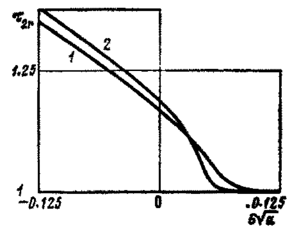

Fig.4

pressure and longitudinal components of the pressure gradient taken from the solution of the equations of the viscous shock layer at high Reynolds numbers ($\mathrm{Re} = 10^6$) for the impermeable body surface ($G = 0$).

Fig.4 shows the relation between the coefficients of friction in the η direction at the surface τ_{2r} of the body, obtained by solving the equations of the shock and boundary layers ($k = 0.4$, $\mathrm{Re} = 5\cdot 10^3$, $\omega = 0.5$ (line 1 for $\eta = 0$ and line 2 for $\eta = 3$) for various values of the injection (suction) parameter $G\sqrt{a}$. It was shown in /1/ for the neighbourhood of the stagnation point that the difference in the coefficient of friction τ_2 in a direction coinciding with the plane containing the cross-section of the body surface with the greatest radius of curvature, can be considerable even at high Reynolds numbers. An analogous result is obtained here also for the whole plane of symmetry. The qualitative behaviour of the dependence of τ_{2r} on the injection parameter remains the same as at the stagnation point, at every point of the plane of symmetry; τ_{2r} increases with increasing suction and decreases with increasing injection, so that at sufficiently large values of the injection parameter the quantity τ_{2r} differs little from unity (i.e. the coefficient of friction obtained from the numerical solution of the boundary layer equation becomes equal to that obtained from the numerical solution of the shock layer equations at high Reynolds numbers). Comparing the coefficients of friction in the ξ direction we see, that the deviation of the ratio τ_{1r} from unity increases as η increases and for $\eta = 3$ ($k = 0.4$, $\mathrm{Re} = 5\cdot 10^3$, $G = 0$) reaches $\sim 7\%$ compared with 2% for $\eta = 0$. This implies that the effect of vortical interaction increases as η increases.

REFERENCES

1. GERSHBEIN E.A. and YUNITSKII S.A., Investigation of a three-dimensional hypersonic viscous shock layer in the neighbourhood of the stagnation point, in the presence of injection or suction. PMM Vol.43, No.5, 1979.
2. EATON R.R. and KAESTNER P.C., Viscous shock layer flow in the windward plane of cones at angle of attack. AIAA Paper, No.134, 1973.
3. EATON R.R. and LARSON D.E., Laminar and turbulent viscous shock layer flow in the symmetry planes of bodies at angle of attack. - AIAA Paper, No.599, 1974.
4. KUMAR A. and GRAVES R.A., Jr. Numerical solution of the viscous hypersonic flow past blunted cones at angle of attack. - AIAA Paper, No.172, 1977.
5. KUMAR A. and GRAVES R.A., Jr. Viscous hypersonic flow past blunted cones at small angles of attack. - AIAA Journal, Vol.15, No.8, 1977.

6. KUMAR A., Low Reynolds number flow past a blunt axisymmetric body at angle of attack. AIAA Journal, Vol.15, No.8, 1977.
7. KARYAKIN V.E. and POPOV F.D., Calculation of the three-dimensional flow of a viscous, heat conducting gas past a blunted body. Zh. vychisl. matem. i matem. fiz., Vol.17, No.6, 1977.
8. GOLOVACHEV YU.P. and KARYAKIN V.E., Three-dimensional non-equilibrium supersonic gas flow past blunted bodies. Numerical Methods in Mechanics of Continua. Collection of articles. Novosibirsk, Izd-e VTs Akad. Nauk SSSR, Vol.11, No.6, 1980.
9. GERSHBEIN E.A., Asymptotic study of the problem of three-dimensional supersonic flow of a viscous gas past blunted bodies with permeable surfaces. In book: Hypersonic Three-dimensional Flows in the Presence of Physico-Chemical Transformations. Moscow, Izd-vo MGU, 1981.
10. GERSHBEIN E.A., SHCHELIN V.S. and YUNITSKII S.A., Numerical and approximate analytic solutions of the equations of a three-dimensional hypersonic viscous shock layer at moderately small Reynolds numbers. In book: Three-dimensional Hypersonic Flows in the Presence of Physico-Chemical Transformations. Moscow, Izd-vo MGU, 1981.
11. GERSHBEIN E.A., On the theory of three-dimensional hypersonic viscous gas flow past blunted bodies in the presence of injection. In book: Some Problems in the Mechanics of Continua. Moscow, Izd-vo MGU, 1978.
12. GERSHBEIN E.A., Theory of the hypersonic viscous shock layer at high Reynolds numbers and intensive injection of foreign gases. PMM Vol.38, No.6, 1974.
13. TIRSKII G.A., On the theory of the hypersonic flow of a chemically reactive multicomponent gas past plane and axisymmetric blunted bodied, with injection. Nauch. tr. In-ta mekhan. MGU, No.39, 1975.
14. BRYKINA I.G., GERSHBEIN E.A. and PEIGIN S.V., Three-dimensional laminar boundary layer on a permeable surface near the plane of symmetry. Izv. Akad. Nauk SSSR, MZhG, No.5, 1980.
15. PETUKHOV I.V., Numerical computation of two-dimensional flows in a boundary layer. In book: Numerical Methods of Solving Differential and Integral Equation, and Quadrature Formulas. Moscow, Nauka, 1964.

Translated by L.K.

PMM U.S.S.R., Vol.48, No.5, pp. 562-566, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON FREE PERTURBATIONS IN HYPERSONIC LAMINAR FLOW BEHIND A PROFILE*

S.V. MANUILOVICH

Plane-parallel laminar hypersonic flow at large distances behind a wing of infinite span is considered. Non-symmetric free perturbations in the basic flow, described in terms of a blast analogy, are studied. The motion of the gas obeys the Navier-Stokes equations and is specified using two-term asymptotic representations. The symmetric and antisymmetric perturbations of the blast solution have an oscillatory form, with amplitude and frequency decaying in the downstream direction.

1. Formulation of the problem. We shall study a plane parallel flow of a hypersonic $p_\infty = 0$ real gas past a profile. The viscosity λ and thermal conductivity k are assumed to be proportional to the specific enthalpy, and we denote the corresponding proportionality coefficients by λ_0, k_0. The ratio $\varkappa$ of the specific heats C_p and C_V will be assumed to be constant and to satisfy the inequality $1 < \varkappa < 2$. We shall use the density ρ_∞ of the incoming flow, its velocity U_∞ and the coefficient λ_0 as the basic unit measures. The Prandtl number $\Pr = C_p\lambda_0/k_0$.

We introduce the notation $1 + v_x$, v_y for the components of the velocity vector along the x, y axes of a Cartesian system of coordinates whose origin coincides with the streamlined profile whose abscissa axis coincides with the direction of the incoming flow. We denote the pressure, density and specific enthalpy by p, ρ, w respectively. In describing the motion of gas we shall use the system of Navier-Stokes equations and the Mises x, Ψ variables.

The principal terms of the asymptotic expansions $x \to \infty$ describing the laminar hypersonic flow of a viscous, heat conducting gas behind a body of finite dimensions, were obtained in /1/. The solution constructed there is symmetrical about the streamline $\Psi = 0$ and includes

*Prikl.Matem.Mekhan., 48,5,776-781, 1984